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ESTIMATION AND CONTROL OF A MACROECONOMIC MODEL WITH RATIONAL EXPECTATIONS

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The paper investigates an econometric method for selecting macroeconomic policy rules when expectations are formed rationally. A simple econometric model of the U.S. is estimated subject to a set of rational expectations restrictions using a minimum distance estimation technique. The estimated model is then used to calculate optimal monetary policy rules to stabilize fluctuations in output and inflation, and to derive a long run tradeoff between price stability and output stability which incorporates the rationally formed expectations. The optimal tradeoff curve is compared with actual U.S. price and output stability and with the results of a monetary policy rule with a constant growth rate of the money supply.

1. INTRODUCTION

A TROUBLESOME SHORTCOMING with contemporary methods of quantitative macroeconomic policy is the failure to take full account of business and consumer reactions to the policies formulated. This problem is characteristic of both policy simulation and formal optimal control techniques, each of which are based on reduced form econometric models in which output and price expectations are formed by fixed coefficient distributed lag structures. Since these lag structures show no direct relationship to government policy, the mechanisms generating expectations are in general inconsistent with the expectations of firms and consumers who are aware of this policy.[2]

Finding empirical methods to deal with this problem is potentially important for a number of reasons. The social welfare gains expected from plans which rely on unresponsive expectations are likely to be significantly cut short, and perhaps made perversely negative, as people learn about policy through observation. Proper policy formulation therefore requires either the difficult task of modelling how people learn about unannounced plans,[3] or the apparently easier task of publicly announcing plans, assuming that these will be incorporated in peoples' information sets. Announcement of policy plans may also have a direct

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[2] This problem was emphasized in the important critique by Lucas [**7**] based on a rational expectations analysis. One reason for the lack of attempts to revise the methodology along the lines suggested by Lucas might be the related (though quite different) critique emphasized by Sargent and Wallace [**17**], that monetary policy does not affect output at all with rational expectations. The methodological approach suggested in this paper is relevant because the Sargent–Wallace proposition does not hold in our model.

[3] One approach to modelling how people learn about policy is examined in Taylor [**20**].

stabilization effect by creating an atmosphere in which business and labor can avoid inflationary wage and price decisions: the removal of some uncertainty regarding inflation may reduce the incentives for inflationary bias in wage and price settlements. A number of central banks have already begun to announce their near term monetary growth plans, and other central banks may soon follow suit. If quantitative macroeconomic policy methods are to be useful in such an environment, they must be able to incorporate the effects of this public announcement. Despite this apparent importance, however, there has been little empirical work on the problem.

The object of this paper is to investigate an empirical method to take account of these expectation effects. The method involves estimating an econometric model in which expectations are rational, and subsequently using this estimated model to calculate optimal monetary control rules. The rational expectations approach constrains the expectation variables to be consistent with the announced policies, and therefore is a way to avoid the problem mentioned above. The model estimated here is highly aggregated, but has the advantage of permitting concentration on the technical problems of estimation and control with rational expectations. Section 2 introduces the basic assumptions of the model and shows how it can be reduced to a policy-invariant form suitable for estimation and control. In Section 3 the model is estimated using U.S. quarterly data from 1953 through 1975, employing a minimum distance estimation technique which takes account of the restrictions imposed by the rational expectations. In Section 4 the estimated model is used to calculate an optimal monetary control rule which incorporates the rational expectations restrictions. The main result of this policy calculation is an empirical efficiency locus representing the best long-term tradeoff between output stability and inflation stability. This efficiency locus is measured in terms of the fluctuations of output and inflation about target values, and is compared with the actual performance of the U.S. economy during the 1953–75 period and with the performance of a fixed monetary growth rule.

2. THE STRUCTURE OF THE MODEL

The model we shall work with is a very small one in which all consumption and investment demands have been reduced to a single aggregate demand equation, and all wage and price decisions have been reduced to a single aggregate price determination equation. For the purpose of finding policy rules to stabilize output and inflation, such an aggregated model is sufficient if its parameters are policy-invariant; that is, if the model is *structural* in the sense that the parameters can be treated as fixed over the relevant range of potential changes in the policy rule.[4]

Some of the parameters of the model are made structural through the use of rational expectations. For example, the parameters in the expected inflation and expected output equations are structural because these equations are consistent with the overall model and hence with the behavior of policy. If adaptive

[4] Sims [**18**] defines structure this way.

expectations were used, then the coefficients of expectation would not be policy-invariant. Other parameters of the model are made structural by assumption. For example, the accelerator coefficient in the investment function is assumed to be unaffected by changes in policy over a certain range. Hence, this paper deals explicitly with one common type of parameter variation problem—that caused by *ad hoc* treatment of expectations—assuming that other types of potential parameter variations are relatively small.[5]

The model is assumed to take the following form:

$$y_t = \beta_1 y_{t-1} + \beta_2 y_{t-2} + \beta_3(m_t - p_t) + \beta_4(m_{t-1} - p_{t-1}) + \beta_5 \hat{\pi}_t + \beta_6 t + \beta_0 + u_t, \tag{1}$$

$$\pi_t = \pi_{t-1} + \gamma_1 \hat{y}_t + \gamma_0 + v_t, \tag{2}$$

$$u_t = \eta_t - \theta_1 \varepsilon_{t-1}, \tag{3}$$

$$v_t = \varepsilon_t - \theta_2 \varepsilon_{t-1}, \tag{4}$$

where y_t is the log of real expenditures measured as a deviation from trend, m_t is the log of money balances during period t, p_t is the log of aggregate price level prevailing during period t, π_t is the rate of inflation defined as $p_{t+1} - p_t$, $\hat{\pi}_t$ is the conditional expectation of π_t given information through period $t-1$, $\hat{y}_t$ is the conditional expectation of y_t given information through period $t-1$, and η_t and ε_t are random shocks to the output and inflation equations. The random vector (η_t, ε_t) is assumed to be serially uncorrelated with mean 0 and variance-covariance matrix Ω.

Equation (1) is the aggregate demand equation. As with more elaborate econometric models, it can be derived from conventional IS–LM relationships. Aggregate demand consists of consumption, investment, government, and net foreign demand. Each of these in turn may depend on such variables as current and lagged values of income and money balances, nominal interest rates, and the expected rate of inflation. These equations can be reduced to an IS relationship by aggregating and solving for total aggregate demand as a function of the nominal interest rate, the expected inflation rate, and other variables including lags. On the LM side, the demand for real money balances depends, with a distributed lag, on the nominal interest rate and the level of aggregate demand. Solving this LM equation for the interest rate and substituting into the IS equation results in the aggregate demand function considered here.

For simplicity we assume that all these relationships can be approximated by functions which are log-linear in real balances and aggregate output and linear in the expected rate of inflation. Because the focus of this paper is on stabilization policy we abstract from long run growth considerations by measuring output as a deviation from trend. The time trend is included in the IS equation to allow for long run secular trends in the money demand function and in the components of aggregate demand.

[5] It should be emphasized that, without some assumptions, it is impossible to prove whether this model, or another, is structural in the sense used here. See Sargent [**15**].

The two lagged values of output are sufficient to capture multiplier-accelerator effects, but may also represent other sources of persistence. One theoretical reason for the lagged value of real money balances is the partial adjustment of these balances to changes in interest rates and income. Since this partial adjustment can be represented by a lagged value of real balances in the money demand equation, we would expect that β_4 should be opposite in sign to β_3 and less in absolute value. Alternatively, real balances may have a direct impact on expenditures which operates with a lag. We will assume throughout the analysis which follows that the β coefficients in equation (1) are invariant to changes in the process generating the policy variable.

Equation (2) is the price determination equation. Note that with π_t defined as $p_{t+1}-p_t$, equation (2) implicitly describes how p_{t+1} is set. Since π_t is a function of v_t (as well as predetermined variables), p_{t+1} is a function of variables with subscripts no greater than t. Likewise p_t is a function of variables before t, and is therefore predetermined at time t. This predeterminacy of p_t is important for what follows.

The rationale for equation (2) is that prices and wages (with a markup to prices) are set in advance of the periods during which they apply (as was assumed by Phelps and Taylor [**12**]). Moreover these prices and wages are not set by all firms simultaneously, and in addition are maintained (on the basis of long term profit considerations) for more than one period. Hence, price and wage decisions are staggered and the multiperiod contracts overlap each other. Equation (2) is meant to approximate such an economy. At any point in time some (but not all) firms will be adjusting their wages and prices and must therefore take into account not only the expected tightness in their market (represented by $\hat{y}_t$ in the aggregate) but also the most recent price and wage decisions of other firms (represented by π_{t-1}). In particular prices will increase more rapidly than π_{t-1} if markets are expected to be tighter than average ($\hat{y}_t > 0$). A more explicit derivation of such an equation based on a simple two period model of staggered pricing is given in Appendix I, but the important aspect of the equation is that the purpose of the lagged inflation rate is not to represent an expectation of π_t, as it might in an expectations augmented Phillips curve. Rather π_{t-1} represents the fact that price and wage decisions of some firms are given at the time that other firms are setting prices and wages, and consequently the current decisions must be made relative to those predetermined values. In fact, the expectation of π_t involves all variables in the model—not just π_{t-1}.

Equation (2) has an important characteristic which is common to most rational expectations models whether based on sticky or flexible prices: it is perfectly accelerationist. In other words there is no way that output can be raised *permanently* above its secular trend growth rate without *accelerating* rates of inflation. In the long run the Phillips curve is vertical, though in the short run (with π_{t-1} predetermined) it is not. Note that by entering a constant term in equation (2) we allow for the fact that the zero change inflation point ($\Delta\pi_i = 0$) may not occur where output equals its estimated secular trend.

Equations (3) and (4) describe the stochastic structure of the random shocks. The shock to the inflation equation has a first order moving average form. This specification allows a fraction θ_2 of a given shock to the inflation rate to be transitory with only $1-\theta_2$ of the shock persisting into the subsequent period. In terms of the staggered pricing model, one economic interpretation of this error structure is that firms realize that there are some nonrecurrent errors or mistakes in the index of other firms' prices; these should not be fully incorporated into their own prices.

The inclusion of two lagged dependent variables in equation (1) leaves very little identifiable serial correlation in the error term u_t. However, the presence of *real* money balances in this equation suggests that the lagged shock from the price equation should be included in the error structure of u_t. A nonrecurrent shock to the price level will change real balances in equation (1) as much as a recurrent shock. But the first type of shock will have a much smaller effect on aggregate demand. Adding the lagged price shock to the equation will allow for this differential effect. Hence, the rationale for equation (3).

For estimation and control this model must be written in a form which does not depend on the unobservable expectations variables $\hat{\pi}_t$ and $\hat{y}_t$. Such a form can be obtained by solving for $\hat{\pi}_t$ and $\hat{y}_t$ in terms of the predetermined variables at time $t-1$ (expectations are conditional on period $t-1$ information), and substituting these solutions into (1) and (2). That the model does not contain expectations of π_{t+i} and y_{t+i} for $i>0$, is a useful simplification. If these multiperiod expectations did appear in the model, then certain complications involving stability or nonuniqueness questions could arise (see Taylor [**21**], for example). Nevertheless, the estimation and control procedures discussed below could be implemented—with suitable modifications—if multiperiod expectations appeared.[6] It should also be emphasized that $\hat{\pi}_t$ and $\hat{y}_t$ are expectations of "future variables" since the conditioning date is $t-1$. The simplification arises from the omission of forward difference equations in the expectations, rather than from the omission of forward expectations.

In order to solve for $\hat{\pi}_t$ and $\hat{y}_t$ recall that p_t is predetermined at time t. We will assume that the money supply m_t is also predetermined so that the conditional expectation of m_t given information through period $t-1$ is equal to m_t itself. This would be the case, for example, if the policy procedure for determining m_t as a function of past observable information was fairly accurately known during the estimation period. (This does not necessarily imply that policy was determined by a constant parameter feedback rule.) We will not directly estimate a policy function for m_t because we can obtain consistent estimates of the structural parameters of (1) and (2) without such estimation. If the policy function is not misspecified, then joint estimation of an m_t equation may increase the efficiency of

[6] The main modification for *estimation* when multiperiod forecasts appear would involve some procedure to generate conditions for solving the forward difference equation in the conditional forecasts. For example, since near term expectations would depend on longer term expectations of future policy variables, a policy function for the money supply should be specified and estimated. See Sargent [**17**]. The assumption used in this paper is convenient because it avoids the problem of specifying and estimating a policy function, but similar nonlinear estimation techniques could be used once a policy function was specified. The required modifications for *control* calculations when multiperiod forecasts appear are outlined in footnote 13 below.

the estimates. However a misspecified policy function could seriously bias the structural estimates of (1) and (2). In this case a robust estimator seems preferable despite a possible loss in efficiency.

With m_t and p_t predetermined, both these variables can be treated as part of the information set at period $t-1$. Therefore, taking conditional expectations in (1) and (2) using (3) and (4), we have

$$\hat{y}_t = \beta_1 y_{t-1} + \beta_2 y_{t-2} + \beta_3(m_t - p_t) + \beta_4(m_{t-1} + p_{t-1}) + \beta_5 \hat{\pi}_t + \beta_6 t + \beta_0 - \theta_1 \varepsilon_{t-1}, \tag{5}$$

$$\hat{\pi}_t = \pi_{t-1} + \gamma_1 \hat{y}_t + \gamma_0 - \theta_2 \varepsilon_{t-1}. \tag{6}$$

Solving these for $\hat{\pi}_t$ and $\hat{y}_t$ and substituting these solution values into (1) and (2) gives

$$y_t = a[\beta_1 y_{t-1} + \beta_2 y_{t-2} + \beta_3(m_t - p_t) + \beta_4(m_{t-1} - p_{t-1}) + \beta_5 \pi_{t-1} + \beta_6 t + \beta_5 \gamma_0 + \beta_0 - (\beta_5 \theta_2 + \theta_1)\varepsilon_{t-1}] + \eta_t, \tag{7}$$

$$\pi_t = a[\gamma_1(\beta_1 y_{t-1} + \beta_2 y_{t-2} + \beta_3(m_t - p_t) + \beta_4(m_{t-1} - p_{t-1})) + \pi_{t-1} + \gamma_1 \beta_6 t + \gamma_1 \beta_0 + \gamma_0 - (\gamma_1 \theta_1 + \theta_2)\varepsilon_{t-1}] + \varepsilon_t, \tag{8}$$

where $a = (1 - \beta_5 \gamma_1)^{-1}$.

(It should be noted that this solution procedure can be performed using matrix notation by defining the vector $z_t = (y_t, \pi_t)'$ as in Section 4 below and solving for $\hat{z}_t$. Hence the derivation of the form (7) and (8) can easily be generalized to higher order systems.)

The reduced form equations (7) and (8) are suitable for estimation and optimal policy calculation. The rational expectations assumption has placed restrictions on the coefficients of these two equations: the 16 coefficients of the predetermined variables (including the lagged disturbance) depend on the 11 unknown parameters in the structural model. Hence, the coefficients of this reduced form are policy-invariant since the parameters of the structural model are. This policy-invariance would not hold if $\hat{\pi}_t$ and $\hat{y}_t$ were assumed to be generated by adaptive expectations, for then the coefficients of expectation in the adaptive formulas would change when the policy rule changed. This in turn would alter the coefficients of the reduced form for $\hat{\pi}_t$ and $\hat{y}_t$.

3. ESTIMATION

Equations (7) and (8) form a vector autoregressive moving-average model with restrictions on the parameters. These restrictions must be satisfied if expectations are to be consistent with the model and with the effects of economic policy. Hence, if the model is to be used for policy purposes—which is our intention here—its parameters should be estimated using a technique which takes account of these restrictions. Constraining the model should also improve the statistical efficiency of the estimators but the more important reason is to insure that expectations will be consistent with economic policy.

In order to estimate (7) and (8) subject to the stated restrictions we use a minimum distance estimator discussed by Malinvaud [**8**]. Writing the model in vector form we have

(9) $$z_t = A(\alpha)x_t + w_t, \qquad w_t = e_t - \theta e_{t-1},$$

where

$$z_t = (y_t, \pi_t)',$$

$$x_t = (y_{t-1}, y_{t-2}, m_t - p_t, m_{t-1} - p_{t-1}, \pi_{t-1}, t, 1)',$$

$$e_t = (\eta_t, \varepsilon_t)',$$

$$\alpha = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \gamma_0, \gamma_1),$$

$$A(\alpha) = a\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & (\beta_5\gamma_0 + \beta_0) \\ \gamma_1\beta_1 & \gamma_1\beta_2 & \gamma_1\beta_3 & \gamma_1\beta_4 & 1 & \gamma_1\beta_6 & (\gamma_1\beta_0 + \gamma_0) \end{bmatrix},$$

$$\theta = a\begin{bmatrix} 0 & \beta_5\theta_2 + \theta_1 \\ 0 & \gamma_1\theta_1 + \theta_2 \end{bmatrix},$$

and where $Ee_te_t' = \Omega$ and $Ee_te_s' = 0$ for $t \neq s$. From equations (7) and (8), $a = (1 - \beta_5\gamma_1)^{-1}$.

The minimum distance estimator (MDE) for this model is obtained by minimizing

(10) $$\sum_{t=1}^{T} e_t'Se_t$$

for some positive definite matrix S. As described in more detail in Appendix II, we iterate the MDE by setting S equal to $(\Sigma_{t=1}^{T} \hat{e}_t\hat{e}_t')^{-1}$ where the $\hat{e}_t$ are the estimated residuals from the previous iteration. Briefly, given values of the serial correlation parameters θ_1 and θ_2, a minimum distance gradient algorithm is used to obtain the minimum with respect to the elements of α. A grid search technique is then used to calculate the smallest of these minima with respect to θ_1 and θ_2.

In a recent paper on the econometric implications of rational expectations Wallis [**23**] considers the estimation of a reduced form system of equations similar to (9) but without serial correlation, and suggests an algorithm to compute the maximum likelihood estimate subject to the rational expectations constraints. Wallis also investigates another estimation problem: that of estimating the parameters of equations which contain more than one current endogenous variable. This problem would correspond to estimating, for example, the marginal propensity to consume out of current income in our model—a parameter which is implicit in the β-coefficients of equation (1). We do not consider such estimation here because our main concern is with calculating optimal control rules, and the reduced form equations (7) and (8) are suitable for this purpose since the parameters are policy-invariant.

The model was estimated using aggregate U.S. quarterly data over the period from 1953:I through 1975:IV. The particular series used for y_t, m_t, and p_t are the deviations of the log of real GNP from the log of potential GNP, the log of M_1 and the log of the GNP deflator, respectively (all seasonally adjusted). The potential GNP series is the recently revised estimate of the Council of Economic Advisers and the other series are taken from the NBER data base. They incorporate the 1976 NIPA revisions.

The parameter estimates and an estimate of the variance-covariance matrix[7] of these estimates are reported in Table I. The estimates of β_1 and β_2 indicate that the aggregate output function is stable for fixed values of the other explanatory

TABLE I

MINIMUM DISTANCE ESTIMATES OF THE MODEL 1953I–1975IV[a]

Output Equation:

$$y_t = \underset{(13.3)}{1.167}y_{t-1} - \underset{(3.6)}{.324}y_{t-2} + \underset{(3.3)}{.578}(m_t - p_t) - \underset{(2.5)}{.484}(m_{t-1} - p_{t-1}) - \underset{(1.4)}{.447}\hat{\pi}_t + \underset{(1.1)}{.0000843}t + \underset{(2.1)}{.0720} + u_t,$$

$$u_t = \eta_t + .38\varepsilon_{t-1}, \quad \hat{\omega}_\eta = .007916.$$

Price Equation:

$$\pi_t = \pi_{t-1} + \underset{(3.1)}{.0180}\hat{y}_t + \underset{(3.0)}{.000515} + v_t,$$

$$v_t = \varepsilon_t - .67\varepsilon_{t-1}, \quad \hat{\omega}_\varepsilon = .003661.$$

Autocorrelations and Cross Correlations of Estimated Residuals:[b]

s	0	1	2	3	4	5	6	7	8
$\rho(\varepsilon_t, \varepsilon_{t-s})$	1.000	−.020	−.023	.077	.038	−.025	.100	.145	−.136
$\rho(\varepsilon_t, \eta_{t-s})$	.012	−.025	−.016	−.009	.050	.071	−.061	.055	.101
$\rho(\eta_t, \varepsilon_{t-s})$	.012	−.139	−.070	−.147	−.125	−.054	−.082	.016	−.139
$\rho(\eta_t, \eta_{t-s})$	1.000	−.002	.034	.115	−.031	−.127	−.031	−.125	−.075

Variance-Covariance Matrix of Estimated Coefficients:[c]

β_1	β_2	β_3	β_4	β_5	β_6	β_0	γ_1	γ_0
.77(2)	−.68(2)	−.39(2)	.31(2)	−.10(2)	.16(5)	.59(3)	.41(5)	.15(6)
	.81(2)	.54(2)	−.61(2)	.42(2)	.38(6)	.51(3)	.43(4)	.71(6)
		.31(1)	−.33(1)	.39(1)	−.19(5)	.20(2)	.17(4)	.31(6)
			.38(1)	−.47(1)	−.58(6)	−.36(2)	−.11(4)	−.25(6)
				.96(1)	−.32(5)	−.56(2)	.31(4)	.76(6)
					.61(8)	.18(5)	−.10(7)	−.26(10)
						.12(2)	−.39(5)	−.58(8)
							.34(4)	.65(6)
								.29(7)

[a] The symbols are defined in the text. Absolute asymptotic t-ratios are printed below the estimated coefficients. The estimated standard errors of the equations are denoted by $\hat{\omega}_\eta$ and $\hat{\omega}_\varepsilon$.

[b] The correlations involving lags of s periods are calculated from the estimated residuals over the sample period 1953III + s through 1975IV.

[c] The numbers in parentheses represent the negative power of 10. Also see footnote 6 in the text concerning the interpretation of this variance-covariance matrix.

[7] The estimated variance-covariance matrix of $\hat{\alpha}$ is computed from the derivatives of (10) with respect to the elements of α. A better estimate of the variance-covariance matrix would take into account the possible correlation between these estimates and the estimates of θ_1 and θ_2, but such an estimate is not easily obtainable with the estimation routines used here. The estimated variance-covariance matrix reported here is conditional on $(\hat{\theta}_1, \hat{\theta}_2)$ and is likely to understate the standard errors.

variables. Writing the terms $1.17y_{t-1}-.32y_{t-2}$ as $.85y_{t-1}+.32(y_{t-1}-y_{t-2})$ shows the magnitude of the "acceleration component" which is added to the first order autoregression. The estimated coefficients of each of the real balance terms are significantly different from zero, as is their sum $\beta_3+\beta_4$. The coefficient of β_4 is negative with absolute value less than β_3, which is consistent with a partial adjustment hypothesis on the money demand equation as discussed in Section 2. The coefficient of the expected inflation rate is negative but not very significant. This sign is opposite to what one would expect on intertemporal substitution grounds—a higher price of future goods relative to current goods should stimulate expenditures. One explanation for the negative sign is that the income effect of a higher expected future price level dominates the substitution effect. Another is that higher expected inflation creates uncertainty which depresses expenditures.[8]

The coefficient of the excess aggregate demand variable in the inflation equation has a positive sign which is in accord with the basic assumption of the model: excess aggregate demand increases inflation. The intercept term in the equation is positive and significantly different from zero which indicates that inflation will be increasing when the economy is operating at the current estimate of potential GNP. The non-accelerating inflation point occurs at a GNP gap of about 2.9 per cent (that is $\Delta\pi_t=0$ when $y_t=-.029$).

The estimated coefficient of .0180 in the price equation indicates that inflation will be reduced by .29 percentage points (at annual rates) for each year that GNP is 1 per cent below the non-accelerating inflation point ($.018\times4\times4=.288$). Using an Okun's law multiplier of 3, this translates into a .9 percentage point drop in inflation for each year that the unemployment rate is 1 percentage point above the non-accelerating inflation rate. For example, in 1975 the GNP gap averaged about 8.7 per cent, or 5.8 per cent above the non-accelerating inflation value. According to the estimate in Table I, this had the effect of reducing the rate of inflation by about 1.7 percentage points during the year.[9]

With regard to the error structure reported in Table I, the first order moving average parameter θ_2 indicates that on average 67 per cent of any shock to the

[8] The optimal control rules reported below do not appear to attempt to "exploit" the presence of the expected inflation rate in the IS equation, unless social preferences place only a very small weight on inflation fluctuations. In other words the estimated policy rules are likely to be robust to errors in estimating β_5. On the other hand, the policy rules are very sensitive to γ_1, and the policy problem would have little meaning if $\hat{\gamma}_1$ were the wrong sign. Such robustness considerations are useful for determining how appropriate optimal control techniques are when parameters are subject to estimation errors.

[9] As a general test of the specification of the model and the constraints imposed by the rational expectations, we also estimated the reduced form equation (9) over the same sample period without constraints. An approximate test of the model can then be obtained from the constrained and unconstrained estimates of the variance-covariance matrix Ω of the residuals e_t. Under the assumption that our iterated estimate of Ω (that is, the iteration of S described in the text), converges to the maximum likelihood estimate under normality (conditional on the initial value of the disturbances) the statistic $T[\log|\hat{\Omega}|-\log|\hat{\Omega}_u|]$ has an asymptotic χ^2 distribution with degrees of freedom equal to the number of constraints (5 in this case), where $\hat{\Omega}_u$ is the unconstrained estimate and $\hat{\Omega}$ is the estimate reported in Table I. The value of this statistic is 12.8 which has a marginal significance level of 2.5 per cent. Hence, if one takes the specification of the model as a maintained hypothesis, then the constraints imposed by the rational expectations are not strongly rejected by the data.

inflation equation is temporary, disappearing in the following period. Further, the negative sign of θ_1 implies that nominal balances do not fully adjust to every shock in the price level. This vector moving average error formulation leaves little serial correlation, as is indicated by the estimates of the autocorrelation and cross-correlation functions reported in Table I. (Note, however, that the standard errors of these correlations are likely to be somewhat less under the null hypothesis of no correlation, than they would be if calculated from the unobservable ε_t and η_t rather than from the estimated $\hat{\varepsilon}_t$ and $\hat{\eta}_t$.) Finally, there is very little contemporaneous correlation between ε_t and η_t.

4. DETERMINATION OF THE OPTIMAL POLICY RULES

Because the parameters of equations (7) and (8) are invariant to the mechanism generating the money supply and because the level of money balances appears explicitly, these equations are suitable for calculating monetary feedback rules using optimal control techniques. In order to obtain empirical specifications for such feedback rules we will treat the estimated values of the parameters of (7) and (8) as equal to their true values. This certainty equivalence approach does not deal explicitly with the joint aspects of estimation and control, but has been found to give good results, at least for large sample sizes,[10] and is frequently used in econometric applications of optimal control theory.

The role of monetary policy in this model is to reduce the fluctuations of real output and inflation about average target levels. A logical target for output is the non-accelerating inflation level of output given by the estimated values of equation (2). Attempts to achieve an average output level higher than this value ($y_t = -.029$) will result in constantly accelerating rates of inflation and would not therefore be consistent with any reasonable objective for inflation. Determining a target level for inflation is more troublesome, however, and would involve a welfare analysis which considers the benefits and costs of alternative average levels of inflation. In order to focus on the stabilization problem we will assume that such an analysis has been completed and that the optimal target rate of inflation is therefore given.

Let y^* and π^* represent these target levels for output and inflation. A loss function which measures the weighted cost of fluctuation about these target levels is given at any point in time by

$$\lambda(y_t - y^*)^2 + (1-\lambda)(\pi_t - \pi^*)^2 \tag{11}$$

where $0 \leqslant \lambda \leqslant 1$. We will focus primarily on finding monetary feedback rules to minimize the expected value of this loss function for the steady state distribution of y_t and π_t. This is equivalent to finding a feedback rule to minimize the expected value of an undiscounted sum of such losses over an infinite time horizon.[11]

[10] See Taylor [**19**] for an analysis of the large sample results in a simple regression model. In using such an argument for the model considered here, we are implicitly assuming that these results can be generalized, though no formal proof is yet available.

[11] Such an undiscounted sum can be normalized so that it converges using a stochastic version of the Ramsey deviation from bliss approach.

In order to describe the optimization procedure we introduce a matrix notation which summarizes the autoregressive and the moving average dynamics as well as the impact of the money supply on these dynamics.[12] Let d_t be the deviation of the log of real money balances from some trend; that is

$$d_t = m_t - p_t - \delta_1 t - \delta_0. \tag{12}$$

Then, by replacing $m_t - p_t$ with d_t in (7) and (8), these equations can be centered on y^* and π^*, and the constant and time trends can be omitted. That is, (7) and (8) can be written as

$$Y_t = BY_{t-1} + cd_t + r_t \tag{13}$$

where

$$Y_t = (y_t, y_{t-1}, d_t, \pi_t, \varepsilon_t)',$$

$$r_t = (\eta_t, 0, 0, \varepsilon_t, \varepsilon_t)',$$

$$c = a(\beta_3, 0, a^{-1}, \gamma_1\beta_3, 0)',$$

$$B = a\begin{bmatrix} \beta_1 & \beta_2 & \beta_4 & \beta_5 & (\beta_5\theta_2 + \theta_1) \\ a^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \gamma_1\beta_1 & \gamma_1\beta_2 & \gamma_1\beta_2 & 1 & (\gamma_1\theta_1 + \theta_2) \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and where y_t and π_t now represent *deviations* from y^* and π^*. In terms of this notation the loss function can be written as

$$Y_t' \Lambda Y_t \tag{14}$$

where Λ is a square weighting matrix with the first diagonal element equal to λ, the fourth diagonal element equal to $(1-\lambda)$, and the remaining elements equal to zero.

With the price level p_t predetermined, the real money balance term d_t can be set at any desired level by the monetary authorities. Consequently d_t can serve as a control variable, and the optimal control problem is to find a feedback rule for d_t to minimize the expected value of the loss function (14) subject to the stochastic dynamics in (13). Hence, the model is now in a form to which existing optimal control procedures can be applied directly (see Chow [**4**]). We will consider feedback rules of the form

$$\begin{aligned} d_t &= g_1 y_{t-1} + g_2 y_{t-2} + g_3 d_{t-1} + g_4 \pi_{t-1} + g_5 \varepsilon_{t-1} \\ &= gY_{t-1}. \end{aligned} \tag{15}$$

Thus, real balances are set according to the most recent observation on the state vector Y_{t-1}. Note that although real balances are predetermined, they are clearly

[12] The procedure used here for dealing with moving average disturbances in control problems by adding these disturbances to the state vector was suggested by Pagan [**19**].

not exogenous. The actual stochastic behavior of real balances will depend on the interaction of the policy rule with the structural distributions of output and prices.

Using optimal control techniques (see Chow [**4**, p. 170]), the value of the vector g which minimizes the expected value of (14) in the steady state is given by

$$g = -(c'Hc)^{-1}c'HB \tag{16}$$

where the matrix H is the solution of the equations

$$H = \Lambda + (B + cg)'H(B + cg). \tag{17}$$

Given the estimated values of the parameters in B and c, we can determine the matrix H and the feedback vector g for any value of λ. The matrix H can be calculated iteratively by computing successive approximations $H_{i+1} = \Lambda + D'H_iD$ where $D = (B + cg)$, starting from some initial approximation H_0 (see Anderson [**2**, p. 182]). The matrix Λ is a good initial value for this iterative procedure.[13]

Before reporting the results of the optimal control calculation, some discussion of recent research by Calvo [**3**], Kydland and Prescott [**6**], and Prescott [**14**] on the problem of time inconsistency is in order. This research has shown that if expectations are rational, then conventional optimal control techniques may be inappropriate because policymakers will have incentive to change their original plan at a later date, when desired economic behavior—partially motivated by anticipations of the original plan—has been achieved. Given this potential inconsistency of optimal policies, an alternative approach would be to forgo optimal policies, and design policies which are consistent. Such consistent policies are analogous to noncooperative solutions in game theory, and in general are suboptimal. In this paper only optimal policies are considered, the hypothesis being that policymakers—with concern about the long run system effects of policy—will not change plans in midstream. In other words we assume that the cooperative solution will be maintained, either because policymakers operate under an incentive system which generates such behavior or because such behavior is legally enforced.[14] Our use of an infinite time horizon with no

[13] A modification of this procedure to deal with a model in which $\hat{y}_{t+1}$ and $\hat{\pi}_{t+i}$ for $i > 0$ appear can be briefly described as follows: By definition, the policy problem described here minimizes E tr $\Lambda\Sigma$ subject to the steady state constraint $\Sigma = V + G'\Sigma G$ where $G \equiv (B + cg)$ is the matrix of lag coefficients in $Y_t = GY_{t-1} + r_t$. Note that the matrix G is a linear function of g; hence (16) is analogous to "generalized linear least squares." If $\hat{y}_{t+1}$ and $\hat{\pi}_{t+i}$ for $i > 0$ appeared in (1) and (2), then, given a policy rule of the form (15) and certain terminal conditions, the reduced form of the system can be shown to be of the same form $Y_t = GY_{t-1} + r_t$ but with the matrix G a *nonlinear* function of the elements of g. Hence, the computation of steady state policy involves the same type of minimization problem as that posed above. However, the nonlinearity of G would involve a more complex computation problem analogous to "generalized nonlinear least squares."

[14] For a further discussion of these issues see the comments by Taylor [**22**] on the paper by Prescott [**14**]. An important practical issue is whether such incentives do exist as part of the political system. It is illustrative to examine the potential for a policy shift in the model of this paper. According to equation (2) the rate of inflation will be reduced if output is *expected* to be below "full employment" output. According to (5) and (6) such a planned recession will be expected by the public if the monetary authorities announce a sufficiently low value of m_t in their plan. Having achieved an expected recession and a corresponding moderation of inflation, the authorities could then fool the public by changing their plan and setting a higher value of m_t to guarantee full employment according to (1). Note that actual m_t appears in (1), while expected m_t appears in (2). The optimization techniques presented in this paper assume that such intentional policy shifts do not occur.

discounting is in keeping with such an assumption. In any case if such an assumption is made, then the optimal control techniques used here are appropriate.

The values of the feedback coefficients for the optimal monetary rules corresponding to several values of λ are given in Table II. The optimal reactions of

TABLE II

OPTIMAL POLICY REACTION FUNCTIONS AND RESULTING OUTPUT-INFLATION VARIATION

Weight on Output Fluctuations (λ)	Reaction Coefficients π_{t-1}	ε_{t-1}	Optimal Standard Deviation of Output (σ_y) (per cent)	Optimal Standard Deviation of Inflation (σ_π) (per cent)
.01	−15.11	9.49	2.14	1.64
.10	−4.32	2.24	1.35	2.04
.20	−2.65	1.11	1.19	2.28
.50	−0.86	−0.09	1.01	2.88
.70	−0.17	−0.55	0.93	3.32
.90	0.29	−0.86	0.85	3.96

Note: Reaction of monetary policy to y_{t-1}, y_{t-2}, and d_{t-1} is identical for all values of λ when there is no cost of control. The coefficients of these variables are −2.02, .56, and .84, respectively. Standard deviation of inflation is given at an annual rate. The reaction coefficients and the standard deviation pairs are computed using the estimated coefficients in Table I and employing an optimal control technique described in the text.

monetary policy to the lagged values of output and the lagged value of real balances are identical for all values of λ. In particular $g_1 = -2.02$, $g_2 = .56$, and $g_3 = .84$ for all λ. Hence, the only difference between feedback rules which are inflation-regarding (small λ) and those which are output-regarding (large λ), is in their reaction to lagged inflation and to the previous price shock (g_4 and g_5). Several optimal values for g_4 and g_5 are listed in the second and third columns of Table II for values of λ ranging from .01 to .90.

When λ is small the optimal policy reacts to increases in inflation above the target level by sharply reducing the growth rate of real balances. However, this deflationary response is softened to the extent that the rise in inflation is expected to be nonrecurrent, as represented by the offsetting positive coefficient of ε_{t-1}. As λ is increased, indicating less concern about fluctuations in inflation, these reaction coefficients move toward zero; in other words monetary policy is more accommodating to changes in the inflation rate. (When λ gets very large the coefficient of π_{t-1} becomes positive, but remains small; policy then attempts to offset the influence of the expected inflation rate in equation (1) in order to stabilize output. Choice of a policy in this range would be very unlikely, however, because of the extraordinarily large fluctuations in inflation; when $\lambda = 1$ the variance of inflation is infinite.)

That the optimal response of policy to lagged values of output and real balances is identical, regardless of the relative concerns about inflation and output, is an important characteristic of the model. The economic reason for the result is that fluctuations in output directly increase fluctuations in inflation through the influence of aggregate demand on prices. In other words policy will reduce the variability of *both* output and inflation by reducing the "own-persistence" of business cycle fluctuations; that is, by offsetting the influence of y_{t-1} and y_{t-2} on y_t. For example, if the inflation rate is currently on target and the economy begins to fall into a recession, then the optimal policy is to stimulate the economy back to full employment as quickly as possible. But, if the inflation rate is above target, then the optimal policy (assuming that the weight on inflation fluctuations is positive) calls for a slower return to full employment. These implications of the optimal control calculation are not inconsistent with many current theories of macroeconomic policy, though these are not usually stated in terms of the variability of output and inflation.

It should be noted that these policy rules do not display instrument instability. Including the policy instrument in the loss function reduces the reaction coefficients, but also detracts from economic performance.

5. THE OUTPUT-INFLATION VARIANCE TRADEOFF

There is no long run tradeoff between the level of output and the level of inflation in this model—the Phillips curve is vertical in the long run. However, there is a long run tradeoff between fluctuations in output and fluctuations in inflation. In other words there is a "second order" Phillips curve which is not vertical in the long run. In order to determine this long run tradeoff, we need the steady state values of $\sigma_y^2 = E(y_t - y^*)^2$ and $\sigma_\pi^2 = E(\pi_t - \pi^*)^2$ corresponding to various values of λ. The graph of σ_y versus σ_π then traces out a minimum variability efficiency locus between output and inflation. This efficiency locus is the tradeoff curve.[15]

For a particular feedback vector g (which is a function of λ), the stochastic behavior of the vector Y_t is described by (13) with $d_t = gY_{t-1}$. Hence the steady-state variance-covariance matrix of Y_t is given by the matrix Σ which satisfies the equations

$$\Sigma = V + (B + cg)'\Sigma(B + cg) \tag{18}$$

where V is the variance-covariance matrix of r_t. Equation (18) is analogous to equation (17) and can be solved by the same iterative procedures described in Section 4. Since Y_t is measured in deviation form, the required values of σ_y^2 and σ_π^2 can be obtained from the first and fourth diagonal elements of the variance-covariance matrix Σ.

The fourth and fifth colums of Table II give several values of σ_y and σ_π calculated according to the above procedure (σ_π has been multiplied by 4 to give annual rates and both standard deviations are stated as per cents). These same

[15] The tradeoff between output and price stability is most easily characterized in terms of the standard derivations of output and inflation in this model. Other characterizations may be more convenient in other models.

values are plotted to trace out an efficiency locus in Figure 1. As one would expect, the tradeoff curve is downward sloping with small values of λ giving points on the upper part of the curve. The minimum value of σ_y is .8 per cent but is not reached

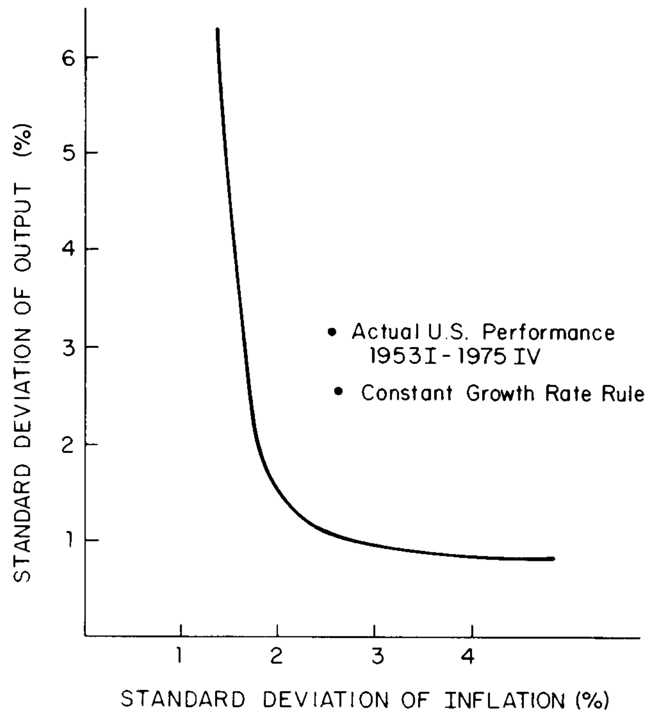

FIGURE 1—Output-inflation variation tradeoff.

SOURCE: See Table 2. The points on this tradeoff curve represent the optimal standard deviations of quarterly output and inflation rates stated at annual rates in per cent. Output is measured as a deviation from a "full-employment" output.

for finite σ_π; the minimum value of σ_π is 1.44 per cent and is reached when σ_y is 6.37 per cent. Hence the tradeoff becomes vertical when output fluctuations reach a standard deviation of slightly over 6 per cent.

A striking characteristic of the tradeoff curve is its sharp curvature: its slope increases from about $-1/4$ to -4 as σ_y increases from 1 to 2 per cent. Hence, only extremely uneven concerns about inflation or unemployment (i.e., only very steep or very flat indifference curves) would lead policymakers to choose a monetary rule which generates output variability outside this 1 to 2 per cent range.

6. EFFICIENT RULES VERSUS ACTUAL U.S. PERFORMANCE AND CONSTANT MONEY GROWTH

It is informative to compare this estimated tradeoff curve with actual U.S. economic performance over the sample period and with the simulations of a constant growth rate rule (CGRR) for the money supply. To determine the actual values of σ_y and σ_π we need target levels for the output gap $(-y^*)$ and the rate of inflation (π^*). For output we use the nonaccelerating inflation point $y^* = -.029$, which is consistent with the model considered here, and not much different from the sample mean $(-.019)$ of y. For the inflation target, we use the sample mean inflation rate of 3.5 per cent, although this probably overstates π^* and hence gives an underestimate of σ_π. The resulting estimated values are $\sigma_y = 3.13$ per cent and $\sigma_\pi = 2.59$ per cent over the 1953I–1975IV period; this pair is shown in Figure 1. It is evident that the actual U.S. economic performance was inefficient during this period according to these criteria, but perhaps not as inefficient as one would have expected. Note that in percentage terms there is more room for reduction in output fluctuations than in inflation fluctuations, if the 3.5 per cent target inflation is reasonable. (Recall that lowering π^* would move the actual performance point to the right, and indicate more potential improvement on the inflation front).

The performance of the rational expectations economy under a CGRR can be determined by substituting the implied real balance feedback coefficients g into equation (18). For a CGRR real balances have an elasticity of -1 with respect to the inflation rate, and an elasticity of 1 with respect to lagged real money balances. Hence, the vector g is equal to $(0, 0, 1, -1, 0)$ when the growth rate of the nominal money supply is constant.[16] This value of g gives $\sigma_y = 2.54$ per cent and $\sigma_\pi = 2.66$ per cent, which is inefficient relative to the estimated tradeoff curve. It is interesting that this simple rule gives an output variance considerably below the actual U.S. performance. It does not quite dominate this performance because the inflation variance is slightly higher. However, if we evaluated U.S. performance at a 3 per cent rather than a 3.5 per cent target inflation rate, then the CGRR would clearly dominate.

7. CONCLUDING REMARKS

The central purpose of this paper has been to present an econometric method for selecting macroeconomic policy when expectations are formed rationally. The method consists of two steps: First, a structural econometric model with rational expectations is estimated using a minimum distance estimation technique. The estimation technique insures that the restrictions imposed on the model by rational expectations are satisfied. Second, this estimated model is used to calculate optimal monetary control rules to stabilize fluctuations in output and inflation. Since the estimated parameters of the model satisfy the rational expectations restrictions, peoples' expectations will be consistent with the policy rule selected. Hence, the method takes account of the reaction of people to expectations of changes in the policy variable as described by the policy rule.

Although the emphasis of the paper is on issues of econometric methodology, a number of results with potential economic policy implications have been derived: (i) Although there is no long run tradeoff between the level of inflation and the

[16] The constant rate of money growth can be absorbed in the δ_1 coefficient of equation (12).

level of output, there does exist a second order Phillips curve tradeoff between fluctuations in output and fluctuations in inflation which is not vertical in the long run. This tradeoff was estimated for the U.S. economy over the 1953–1975 period and is downward sloping: over the relevant range of this curve business cycle fluctuations can be reduced only by increasing the variability of inflation. (ii) As one would expect the optimal monetary policy accommodates increases in inflation when there is great concern with stabilizing output and little concern with fluctuations in inflation. On the other hand, the optimal policy is extremely non-accommodative when fluctuations in inflation are viewed as very harmful. Even in this latter case, however, the optimal policy accommodates non-recurrent shocks to the inflation rate. Reacting too strongly to such temporary shocks is inefficient and leads to an unnecessarily high variability of inflation. (iii) The partial elasticity of the optimal policy rule with respect to deviations of the economy from its potential growth path, is identical regardless of the slope of the output-inflation variance indifference curve. Reducing the "own-persistence" of output is desirable for reducing variability of inflation as well as output. (iv) The actual performance of the U.S. economy of the 1953–1975 period was inefficient relative to the estimated tradeoff. Given the shocks to the economy during this period the standard deviation of output could have been about two percentage points lower for the same variability of inflation. (v) Simulation of a constant growth rate rule (CGRR) for the money supply in the rational expectations model gives a variability of output which is less than the actual U.S. performance over the sample period. If one evaluated U.S. inflation performance using a target inflation rate of 3 per cent, then the CGRR would also give a smaller variability of inflation, and would consequently dominate actual U.S. performance. However, the CGRR is still inefficient relative to the combinations of output and inflation fluctuations that the model indicates are feasible.

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APPENDIX I: Lagged Price Effect due to Staggered Overlapping Contracts

In Section 2 it was argued that the lagged inflation rate on the right-hand side of the price determination equation (2) can be explained by staggered overlapping price and wage contracts at fixed predetermined levels. While some firms are setting prices and wages, other firms will have already made their pricing decisions and these old prices and wages will be maintained at fixed levels during part of the new contract period. Hence, the firms setting prices and wages now will do so relative to the given price and wage decisions of other firms and according to expected demands in their markets.

To illustrate how such pricing behavior can lead to aggregate price equations like (2) we consider in this appendix a very simple model with two types of firms and two period contracts. The model is meant to be suggestive rather than a rigorous derivation of equation (2). Suppose that type 1 firms set prices to take effect at the beginning of odd numbered time periods, and type 2 firms set prices to take effect at the beginning of even numbered periods. For each type of firm the price remains in effect for two periods.[17] Thus, if q_t represents the log of prices set to begin in period t (by type 1 firms if t is odd, and by type 2 firms if t is even) then the log of the geometric aggregate price during period t is

$$p_t = .5(q_{t-1} + q_t). \tag{1.1}$$

Consider a representative type 1 firm deciding what price to set for periods t and $t+1$. The firm knows that for the duration of period t the price of type 1 firms is fixed at q_{t-1} and will change at the end of period t to q_{t+1} which is currently unknown. A reasonable pricing assumption—which is analogous to that proposed by Phelps [**10**] in a nonstaggered model—is that the representative firm sets its price higher than the average price it expects other firms to set, when markets are tight, and conversely sets a relatively low price in slack markets. That is,

$$q_t = .5(q_{t-1} + \hat{q}_{t+1}) + \alpha(\delta_t + \hat{\delta}_{t+1}), \tag{1.2}$$

where the first term on the right-hand side represents the expected average price of the type 2 firms during periods t and $t+1$, and where δ_t is a measure of market excess demand. As in Section 2 the "hat" notation represents conditional expectations given information at time $t-1$.

As an example of the type of aggregate price behavior which is implied by (1.2) suppose that δ_t is a serially uncorrelated random variable with zero mean. Then a stochastic process for q_t which satisfies (1.2) under the rational expectations assumption can be found by substituting the trial solution $q_t = \pi_1 q_{t-1} + \pi_2 q_{t-2} + \alpha\delta_t$ into (1.2) and solving for π_1 and π_2. A solution is $\pi_1 = 2$ and $\pi_2 = -1$, so that[18]

$$q_t = 2q_{t-1} - q_{t-2} + \alpha\delta_t. \tag{1.3}$$

Therefore from (1.1) the aggregate price level p_t follows the second order autoregressive—first order moving average process,

$$p_{t+1} = 2p_t - p_{t-1} + .5\alpha(\delta_{t+1} + \delta_t). \tag{1.4}$$

Or, in terms of the inflation rate,

$$\pi_t = \pi_{t-1} + \alpha\bar{\delta}_t, \tag{1.5}$$

where δ_t is a measure of average excess demand in period t and $t+1$. Equation (1.5) is similar to traditional disequilibrium price adjustment assumptions except that the rate of change in the inflation rate, rather than the rate of change in price, depends on the level of excess demand. But the important result is that lagged prices appear on the right hand side of the equation. In more elaborate models the dynamics will generally be of higher order and will depend on economic policy.[19] Since actual pricing is certainly more elaborate than in this simple model, the aggregate price equation (2) in the text can only serve as an approximation. Note also that the δ_{t+1} term is not explicitly treated in equation (2).

APPENDIX II: USE OF THE MINIMUM DISTANCE ESTIMATOR FOR RATIONAL EXPECTATIONS MODELS

According to the notation of Section 4, the reduced form model we estimate is of the form

$$z_t = A(\alpha)x_t + w_t, \quad w_t = e_t - \theta e_{t-1}, \tag{2.1}$$

where e_t is a serially uncorrelated random vector with mean zero and variance-covariance matrix Ω. Because expectations are assumed to be formed rationally, the 14 elements of the matrix A are restricted in the sense that they are functions of the 9 unknown elements in the parameter vector α. Similar restrictions will be imposed on the reduced form parameter matrix A in other types of rational

[17] This assumption distinguishes this model from that of Fischer[**5**] where two different wage levels are set for the following two periods. Hence in Fischer's model the wage can be set so as to equate expected supply and demand in both periods. Akerlof [**1**] considers an overlapping model similar to the one discussed here.

[18] Some important nonuniqueness problems arise in this type of model and are explored in Phelps [**11**].

[19] Incorporating the dependence of these dynamics on policy is a potentially important extension of the model examined in this paper and is the subject of my own current research.

expectations models. Hence, the notation $A(\alpha)$ is quite general and the following estimation technique is not confined to the rational expectations model considered in this paper.

If w_t were uncorrelated ($\theta = 0$), then the minimum distance estimator (MDE) of α could be obtained by minimizing

$$\sum_{t=1}^{T} (z_t - A(\alpha)x_t)' S(z_t - A(\alpha)x_t) \tag{2.2}$$

with respect to α, for some positive definite matrix S. Malinvaud [8] proposed that the MDE be iterated by setting S to $(\Sigma_{t=1}^{T} \hat{e}_t\hat{e}_t')^{-1}$ at each iteration where the $\hat{e}_t$ are the residuals from the previous iteration, and derived the asymptotic distribution of the estimates; he also suggested that this iterated MDE would converge to the maximum likelihood estimator, calculated as if w_t were normally distributed. Phillips [13] proved that under certain conditions the iterated MDE does converge to this maximum likelihood estimator at least for large sample sizes. Computer routines for calculating the iterated MDE and the asymptotic variance covariance matrix are now widely available. For example TSP (Time Series Processor version 2.7) has such a minimum distance estimator routine which appears to work well in many applications. At least in the case of serially uncorrelated disturbances such routines are therefore readily applicable for estimating reduced forms of rational expectations models.

If θ is not equal to the zero matrix, but the elements of θ are known, then this iterated MDE can be modified to deal with the implied serial correlation. From (2.1) we have that

$$\begin{aligned} e_t &= \sum_{i=1}^{t} \theta^{t-i} w_i + \theta^t e_0 \\ &= \sum_{i=1}^{t} \theta^{t-i} z_i - \sum_{i=1}^{t} \theta^{t-i} A(\alpha) x_i + \theta^t e_0. \end{aligned} \tag{2.3}$$

Given e_0, (2.3) can be used to calculate e_t and the MDE is then obtained by minimizing

$$\sum_{t=1}^{T} e_t' S e_t \tag{2.4}$$

with respect to α. When θ is not known and when there are no restrictions placed on the elements of θ a simple procedure is feasible: for each value of θ in a given region, (2.4) is minimized with respect to α as above. The MDE is then given by the value of θ which gives the smallest value for the minimum of (2.4).

A simple recursive relationship can be used to calculate e_t as a function of the elements of θ. Write $\Sigma_{i=1}^{t} \theta^{t-i} A(\alpha) x_i$ as

$$\sum_{i=1}^{t} (x_i' \otimes \theta^{t-i}) \operatorname{vec} A(\alpha) = X_t^* \operatorname{vec} A(\alpha) \tag{2.5}$$

where $X_t^* = \Sigma_{i=1}^{t} (x_i' \otimes \theta^{t-1})$, and let $z_t^* = \Sigma_{i=1}^{t} \theta^{t-i} z_i$. Then, for $e_0 = 0$,

$$e_t = z_t^* - X_t^* \operatorname{vec} A(\alpha). \tag{2.6}$$

The variables z_t^* and X_t^* are functions of the elements of θ and can be calculated from the relations

$$z_t^* = z_t + \theta z_{t-1}^*, \tag{2.7}$$

$$X_t^* = (x_t' \otimes I) + X_{t-1}^* (I \otimes \theta). \tag{2.8}$$

Hence, using (2.6) and these recursive relations the MDE can be calculated for a given θ with the same algorithms designed for the serially uncorrelated case. In the applications considered here, the lower off-diagonal element of θ is zero; hence the first equation of the transformed model contains all the elements of $A(\alpha)$, while the second contains only the elements in the second row of $A(\alpha)$.

In applying this estimation technique to the model of this paper we took the initial condition $e_0 = 0$. Hence the estimates are conditional at this value, though with 88 observations we would expect that the final estimates are not sensitive to this condition. With only two elements of θ unknown, a two dimensional grid search was used to determine the MDE estimate of θ. (Note that in this model the rational expectations assumption does not put any restrictions on θ. Although β_5 and γ_1 enter into the elements of θ, these are not restrictive since there are exactly two unknown elements in θ and two free parameters θ_1 and θ_2 which do not appear elsewhere in the model.)

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